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Spinning Particles in Spacetimes with Torsion

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A novel analysis of the Mathisson-Papapetrou-Dixon equations is presented employing mathematical tools that do not rely on the torsion free geometries used in previous literature. A system of differential algebraic equations that can be used to describe the motion of spinning particles in an arbitrary geometry is derived. The curvature in these equations can involve non-Riemannian contributions. Subsequently, this particular system of equations can accommodate modification to geodesic motion from both scalar fields and the spin of the particle.

KEY WORDS: Gravity; spinning particles; relativistic dynamics; torsion; Brans-Dicke.

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1. INTRODUCTION

Extended bodies or elementary particles with angular momentum in a background curved spacetime are expected to experience forces and torques. This behaviour can be relevant to many astrophysical phenomena that have been already observed. It is widely suspected that exotic celestial processes such as the gamma-ray bursts, astrophysical jets or X-ray emitters might be the product of the dynamics of electrically charged relativistic matter with angular momentum in the presence of strong electromagnetic and gravitational fields. As a consequence massive test particles with intrinsic spin, moving in strong gravitational fields, are not expected to follow time-like geodesic worldlines (Mohseni *et al.*, 2001).

In principle, one can adopt some basic mechanism for the coupling of charge and angular momentum distributions to such fields and attempt to model the above systems using the relativistic Einstein-Maxwell-Boltzmann equations (Ehlers, 1971). However, it was soon realised that solving those equations for even simple matter models is often impossible and approximation schemes needed to

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be employed. Mathisson, Fock, Papapetrou *et al.* (Mathisson, 1937; Papapetrou, 1951) were the first to study in detail the dynamics of such particles by neglecting self-gravitation and back reaction. Some years later, Dixon used a rationalised multipole expansion technique to further elucidate the previous work (Dixon, 1964, 1965, 1970a,b, 1973, 1974). More specifically, he suggested that a finite collection of mass and charge multipoles in an arbitrary background spacetime could determine the worldline for continuous matter. Although the resulting equations of motion offer a consistent dynamical scheme for the classical behaviour of "spinning matter" they are difficult to solve analytically, even for simple gravitational fields and in the lowest pole-dipole approximation (Tod and Felice, 1976). Up to the dipole approximation the equations of motion of spinning particles are known as the Mathisson-Papapetrou-Dixon (MPD) equations and they encode a spin-curvature coupling that may play an important role in astrophysical systems.

Nevertheless, gravitational theories with spinning matter reside rather unnaturally in the pseudo-Riemannian (i.e. torsion free) spacetime in which the whole Dixon framework was formulated. Furthermore, as we have experienced in the case of the Brans-Dicke theory (Brans and Dicke, 1961), it is not necessary to have spinning sources to accommodate gravitational fields with torsion. As a result, one should consider seriously the possibility that gravitation may have a torsional component that is in principle absent in the pseudo-Riemannian spacetimes adopted in Einstein's theory (Tucker, 2004). The structure of this work involves analysing the motion of spinning test particles in background geometries more general than General Relativity.

In the monopole-dipole approximation the history of a particle with non-zero mass ($m \neq 0$) can be described by a time-like parametrised curve with parameter τ and tangent vector $V(\tau)$. The particle's curve is assumed to be initially future pointing and the dynamics of the spinning particle is also determined by a second time-like vector $P(\tau)$ and a space-like vector $\Sigma(\tau)$ associated with the momentum and the spin respectively. Suppose we have a 4-dim manifold with a metric $\mathbf{g} = \eta_{ab} e^a \otimes e^b$, where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$, and $\{e^a\}$ form a $\mathbf{g}$-orthonormal frame. It is more convenient to relate the elements of the tangent space at each point of the worldline of the particle to elements in the dual space. Therefore, for any vector V in the tangent space we can define its metric dual as $\mathrm{v} = \widetilde{V} = \mathbf{g}(V, -)$. In a similar way, for any covector v in the associated cotangent space we can write $V = \widetilde{\mathrm{v}} = \mathbf{G}(\mathrm{v}, -)$, where $\mathbf{G}$ is the inverse of $\mathbf{g}$.

According to Dixon's formulation, the "intrinsic" spin covector, $l \equiv -\frac{1}{2} \star (u \wedge s)$, can be considered as a distributional approximation to the collective history of matter about the worldline defined by $P \equiv mc^2 U = mc^2 \widetilde{u}$ and $\Sigma \equiv \frac{\widetilde{l}}{m} \equiv \widetilde{\sigma}$ in any background metric (Hodge map, $\star$, is defined by the volume element $\Omega = e^0 \wedge e^1 \wedge \wedge e^2 \wedge e^3$ and we can write $\star 1 = \Omega$), where m is the mass scalar, u the velocity 1-form normalised with $\mathbf{g}(\widetilde{u}, \widetilde{u}) = -1$ and σ the radius spin 1-form.

Then, the MPD equations for a tangent vector $V(\tau)$, momentum vector $P(\tau)$ and spin 2-forms $s(\tau)$ given a metric $\mathbf{g}$ are,

$$\dot{P} = \widetilde{i_V f}, \tag{1}$$

$$\dot{s} = 2\widetilde{P} \wedge \widetilde{V}, \tag{2}$$

and

$$i_P s = 0, \tag{3}$$

where $f = -\frac{1}{4} \star (R_{ab} \wedge \star s) e^a \wedge e^b$ is the tidal bivector along the worldline in any local cobasis $\{e^a\}$ and R_{ab} are the curvature 2-forms of ∇ in this cobasis. Also note that for any tensor $Q(\tau)$ along the worldline we write $\dot{Q} \equiv \nabla_V Q$ in terms of a general connection restricted to this curve and i_V denotes interior contraction with V (Mohseni *et al.*, 2001).

In component notation we can write $R_{ab} = \frac{1}{2} R_{abpq} e^p \wedge e^q$ and $s = s_{cd} e^c \wedge e^d$. So,

$$R_{ab} \wedge \star s = \frac{s_{cd}}{2} R_{abpq} [e^p \wedge e^q \wedge \star(e^c \wedge e^d)] = s_{cd} R^{cd}{}_{ab} \star 1, \tag{4}$$

or

$$\star^{-1}(R_{ab} \wedge \star s) = R_{abcd} s^{cd}. \tag{5}$$

Then Eq. (1) becomes,

$$\dot{p}^b = \frac{1}{2} R_{bacd} s^{cd} \mathrm{v}^a. \tag{6}$$

Similarly, Eq. (2) becomes,

$$\dot{s}_{ml} = p_m \mathrm{v}_l - p_l \mathrm{v}_m. \tag{7}$$

Finally, recall that for any vector field X and any form Φ we can write

$$i_X \star \Phi = \star(\Phi \wedge \widetilde{X}). \tag{8}$$

Also for a p-form α, $\star^{-1}\alpha = -(-1)^{p(n-p)} \star \alpha$, where n is the dimension of the manifold. If we apply these on the r.h.s of (3) we get

$$p \wedge \star s = 0. \tag{9}$$

2. ANALYSIS OF THE MPD EQUATIONS FOR SPINNING PARTICLES

As we have seen we can define the spin 1-form to be

$$l = -\frac{1}{2} \star (u \wedge s), \tag{10}$$

so $i_{\widetilde{p}}s = 0$. Then we can write

$$l = -\frac{1}{2} i_{\widetilde{u}} \star s \quad \Rightarrow \quad i_{\widetilde{u}} l = 0. \tag{11}$$

For generality, we introduce the drive 1-form $\mathcal{K}$ and 2-form $\mathcal{D}$ and consider the equations of motion in a background metric $\mathbf{g} = \eta_{ab} e^a \otimes e^b$. In this case the MPD equations are

$$\dot{p} = i_{\widetilde{\mathrm{v}}} f + \mathcal{K}, \tag{12}$$

with $f = -\frac{1}{4} \star (R_{ab} \wedge \star s) e^a \wedge e^b$,

$$\dot{s} = 2p \wedge \mathrm{v} + \mathcal{D}, \tag{13}$$

and

$$i_{\widetilde{p}} s \equiv p \wedge \star s = 0. \tag{14}$$

Now set $p = m'u$ for a scalar $m' \equiv cm$ so from the last equation we can write $i_{\widetilde{p}}s = i_{m'\widetilde{u}}s = 0 \Rightarrow i_{\widetilde{u}}s = 0$. It is also convenient to adopt a parameterisation for the worldline with a tangent vector v such that $\mathbf{g}(\widetilde{u}, \widetilde{\mathrm{v}}) = -1$ for $\mathbf{g}(\widetilde{u}, \widetilde{u}) = -1$ and we can write

$$i_{\widetilde{u}} \mathrm{v} = -1. \tag{15}$$

Since $\mathbf{g}$ is metric compatible, then

$$\mathbf{g}(\dot{u}, u) = 0 \quad \Rightarrow i_{\widetilde{u}} \dot{u} = 0. \tag{16}$$

From (10) we can write

$$2 \star l \equiv 2 \star^{-1} l = -u \wedge s, \tag{17}$$

and if we apply $i_{\widetilde{u}}$ on both sides of the above equation we get,

$$s = 2 i_{\widetilde{u}} \star l = 2 \star (l \wedge u). \tag{18}$$

It is not difficult to see that,

$$\star s = -l \wedge u \tag{19}$$

Also, from (12) we have

$$\dot{p} \equiv \dot{m}' u + m' \dot{u} = i_{\widetilde{\mathrm{v}}} f + \mathcal{K}. \tag{20}$$

If we apply $i_{\widetilde{u}}$ on both sides of (20) we have,

$$-\dot{m}' = i_{\widetilde{u}} i_{\widetilde{\mathrm{v}}} f + i_{\widetilde{u}} \mathcal{K}. \tag{21}$$

But $i_{\widetilde{\mathrm{v}}} f = -\frac{1}{2} \star (R_{ab} \wedge \star s) \mathrm{v}^a e^b$ and $i_{\widetilde{u}} i_{\widetilde{\mathrm{v}}} f = -\frac{1}{2} \star (R_{ab} \wedge \star s) \mathrm{v}^a u^b$. So, in general we can write

$$\dot{m}' = \frac{1}{2} \star (R_{ab} \wedge \star s) \mathrm{v}^a u^b - i_{\widetilde{u}} \mathcal{K}. \tag{22}$$

Now set $l = m'\sigma$, so $f = \frac{m'}{2} \star (R_{ab} \wedge \sigma \wedge u) e^a \wedge e^b, \mathcal{K} = m'k$ and $\mathcal{D} = m'd$. Then, from (20) we have

$$m'\dot{u} = i_{\tilde{\mathrm{v}}} f + m'k - \dot{m}'u, \tag{23}$$

or

$$\begin{aligned}\dot{u} &= \star(R_{ab} \wedge \sigma \wedge u)\mathrm{v}^a e^b + k + \star(R_{ab} \wedge \sigma \wedge u)\mathrm{v}^a u^b u + u i_{\tilde{u}} k \\ &= (1 + u i_{\tilde{u}})[k + \star(R_{ab} \wedge \sigma \wedge u)\mathrm{v}^a e^b]. \end{aligned} \tag{24}$$

Now, (13) can be written as

$$\dot{s} = 2p \wedge \mathrm{v} + m'd = 2m'u \wedge \mathrm{v} + m'd \tag{25}$$

and

$$\star\dot{s} = 2m' \star (u \wedge \mathrm{v}) + m' \star d. \tag{26}$$

But $\star s = -2m'\sigma \wedge u$. Therefore,

$$-2(m'\sigma \wedge u)^{\cdot} = 2m' \star (u \wedge \mathrm{v}) + m' \star d, \tag{27}$$

or if we expand the l.h.s,

$$\dot{m}'\sigma \wedge u + m'\dot{\sigma} \wedge u + m'\sigma \wedge \dot{u} = -m' \star (u \wedge \mathrm{v}) - \frac{1}{2} m' \star d. \tag{28}$$

We next apply $i_{\tilde{u}}$ to (28), using that $i_{\tilde{u}}\dot{u} = 0$ and $i_{\tilde{u}}\sigma = 0$. The result is

$$\dot{m}'\sigma + m'(i_{\tilde{u}}\dot{\sigma} \wedge u + \dot{\sigma}) = -\frac{1}{2} m' i_{\tilde{u}}(\star d). \tag{29}$$

This equation defines the transformation law for σ. Now multiply both sides by $u\wedge$, so

$$u \wedge (\dot{m'\sigma}) + u \wedge (m'\dot{\sigma}) = -\frac{1}{2} m'u \wedge i_{\tilde{u}}(\star d). \tag{30}$$

But we can use (30) to substitute for $\dot{m}'\sigma \wedge u + m'\dot{\sigma} \wedge u$ in (28) and get

$$\sigma \wedge \dot{u} + \star(u \wedge \mathrm{v}) + \frac{1}{2}(1 + u i_{\tilde{u}}) \star d = 0. \tag{31}$$

Next, we multiply by the Hodge map $\star$ and note that for any $r-$form in the 4-dim spacetime $\star\star\alpha = -(-1)^{r(4-r)}\alpha$. Hence,

$$-\star\sigma \wedge \dot{u} + u \wedge \mathrm{v} - \frac{1}{2} \star (1 + u i_{\tilde{u}}) \star d = 0. \tag{32}$$

Note that this equation involves v in the $u \wedge \mathrm{v}$ term as well as in the $\sigma \wedge \dot{u}$ term since (24) involves v. Now introduce $\widehat{d} = \frac{1}{2} \star (1 + u i_{\tilde{u}}) \star d$ so,

$$\star(\sigma \wedge \dot{u}) - u \wedge \mathrm{v} + \widehat{d} = 0. \tag{33}$$

Applying $i_{\tilde{u}}$ would simply give,

$$\mathrm{v} = u - i_{\tilde{u}}[\star(\sigma \wedge \dot{u})] - i_{\tilde{u}}\widehat{d}. \tag{34}$$

Since the last term is independent of v we write $\Psi \equiv i_{\tilde{u}}\widehat{d}$. Then,

$$\mathrm{v} - u = \star(\sigma \wedge \dot{u} \wedge u) - \Psi. \tag{35}$$

Now write $\hat{k} = (1 + u i_{\tilde{u}})k$ so Eq. (26) becomes,

$$\dot{u} = \hat{\mathrm{k}} + (1 + u i_{\tilde{u}})e^b \star (R_{ab} \wedge \sigma \wedge u)\mathrm{v}^a. \tag{36}$$

Until now we have expressed v in terms of u, σ and $\hat{k}$. Next, we look at the term $\star(\sigma \wedge \dot{u} \wedge u)$ in (35). Using (36) we can write,

$$\dot{u} \wedge u = \hat{k} \wedge u + \star(R_{ab} \wedge \sigma \wedge u)\mathrm{v}^a e^b \wedge u, \tag{37}$$

since $u \wedge u$ is zero. Therefore,

$$\sigma \wedge \dot{u} \wedge u = \sigma \wedge \hat{k} \wedge u + \star(R_{ab} \wedge \sigma \wedge u)\mathrm{v}^a(\sigma \wedge e^b \wedge u) \tag{38}$$

and

$$\star(\sigma \wedge \dot{u} \wedge u) = \Phi + \mathrm{v}^a \star (R_{ab} \wedge \sigma \wedge u) \star (\sigma \wedge e^b \wedge u), \tag{39}$$

where we set $\Phi \equiv \star(\sigma \wedge \hat{\mathrm{k}} \wedge u)$, independent of v. If we substitute (39) in (35) we get,

$$\mathrm{v} - u = \Phi - \Psi + \mathrm{v}^a \star (R_{ab} \wedge \sigma \wedge u) \star (\sigma \wedge e^b \wedge u). \tag{40}$$

So, we only need to solve this last equation for v^c. We apply $i_{\tilde{\sigma}}$ to (40), using that $i_{\tilde{u}}\sigma = i_{\tilde{\sigma}}u = 0$ and $i_{\tilde{\sigma}} \star (\sigma \wedge e^b \wedge u) = \star(\sigma \wedge e^b \wedge u \wedge \sigma) = 0$. Hence,

$$i_{\tilde{\sigma}}\mathrm{v} = i_{\tilde{\sigma}}(\Phi - \Psi). \tag{41}$$

But $i_{\tilde{\sigma}}\Phi = i_{\tilde{\sigma}} \star (\sigma \wedge \mathrm{k} \wedge u) = -(\sigma \wedge \hat{k} \wedge u) = 0$,

$$i_{\tilde{\sigma}}\mathrm{v} = -i_{\tilde{\sigma}}\Psi \equiv -i_{\tilde{\sigma}} i_{\tilde{u}}\widehat{d}. \tag{42}$$

Note that if we apply $i_{\tilde{u}}$ in (40) and using that $i_{\tilde{u}}u = -1$, $i_{\tilde{u}}\mathrm{v} = -1$ we get $i_{\tilde{u}} \star (\sigma \wedge e^b \wedge u) = 0$, $i_{\tilde{u}}\Psi = 0$, $i_{\tilde{u}}\Phi = 0$, so finally $0 = 0$ and nothing new can be provided. So set

$$\beta = \Phi - \Psi, \tag{43}$$

where $\Phi \equiv -i_{\tilde{\sigma}} i_{\tilde{u}} \star \hat{\mathrm{k}}$ and $\Psi \equiv i_{\tilde{u}}\widehat{d}$. Now put $R_{ab} = \frac{1}{2}\epsilon_{ab}^{pq}\widehat{R}_{pq}$, so $\widehat{R}_{cd} = \frac{1}{2}\epsilon_{cd}^{ab}R_{ab}$ and also recall that $R_{ab} = -R_{ba}$. If we substitute all the above in (40) we get

$$\begin{aligned} \mathrm{v} &= u + \beta - \frac{1}{2}\mathrm{v}^a\epsilon_{ab}^{pq} \star (\widehat{R}_{pq} \wedge \sigma \wedge u) \star (\sigma \wedge u \wedge e^b) \\ &= u + \beta - \frac{1}{2}\mathrm{v}^a\epsilon_{ab}^{pq} i_{\tilde{u}} i_{\tilde{\sigma}} \widehat{\widehat{R}}_{pq} \star (\sigma \wedge u \wedge e^b), \end{aligned} \tag{44}$$

where $\widehat{\widehat{R}}_{pq} = \star\widehat{R}_{pq}$. Then, apply i^c to both sides and let the scalar $\Lambda_{pq} = i_{\widetilde{u}} i_{\widetilde{\sigma}} \widehat{\widehat{R}}_{pq} = -\Lambda_{qp}$. After some algebra we get,

$$\mathrm{v}^c - u^c - \beta^c = -\frac{1}{4}\mathrm{v}^a \Lambda_{pq} i_{\widetilde{u}} i_{\widetilde{\sigma}} (e^r \wedge e^s)\epsilon_{ab}^{pq}\epsilon_{rs}^{bc}, \tag{45}$$

where ϵ is the Levi-Civita antisymmetric symbol. In turn, the product $\epsilon_{ab}^{pq}\epsilon_{rs}^{bc}$ can be expressed in terms of the Kronecker symbol δ_b^a

$$\mathrm{v}^c - u^c - \beta^c = -\frac{1}{4}\mathrm{v}^a \Lambda^{rq} i_{\widetilde{u}} i_{\widetilde{\sigma}} (e_r \wedge e_s)\big(2\delta_a^c\delta_p^r\delta_q^s - 2\delta_a^r\delta_p^c\delta_q^s + 2\delta_a^s\delta_p^c\delta_q^r\big) \tag{46}$$

and if we expand we get

$$\mathrm{v}^c - u^c - \beta^c = \frac{1}{2}\mathrm{v}^c \Lambda^{rs} i_{\widetilde{\sigma}} i_{\widetilde{u}} \star (e_r \wedge e_s) - \frac{1}{2}\mathrm{v}^r \Lambda^{cs} i_{\widetilde{\sigma}} i_{\widetilde{u}} \star (\mathrm{v} \wedge e_s)$$
$$+\frac{1}{2}\mathrm{v}^s \Lambda^{cr} i_{\widetilde{\sigma}} i_{\widetilde{u}} \star (e_r \wedge \mathrm{v}). \tag{47}$$

Now set $\Lambda \equiv \frac{1}{2}\Lambda^{rs}(e_r \wedge e_s) = \frac{1}{2}(i_{\widetilde{\sigma}} i_{\widetilde{u}} \widehat{\widehat{R}}_{rs})(e^r \wedge e^s)$, where Λ is a 2-form and note that $\Lambda^{cr} i_{\widetilde{\sigma}} i_{\widetilde{u}} (e_r \wedge \mathrm{v}) = -\Lambda^{cr} i_{\widetilde{\sigma}} i_{\widetilde{u}} (\mathrm{v} \wedge e_r)$. If we use these in (47) we get

$$\mathrm{v}^c - u^c - \beta^c = \mathrm{v}^c i_{\widetilde{\sigma}} i_{\widetilde{u}} \Lambda - \Lambda^{cs} i_{\widetilde{\sigma}} i_{\widetilde{u}} (\mathrm{v} \wedge e_s). \tag{48}$$

It is not difficult to show that

$$i_{\widetilde{\sigma}} i_{\widetilde{u}} (\mathrm{v} \wedge e_s) = -u_s i_{\widetilde{\sigma}} \mathrm{v} - \sigma_s, \tag{49}$$

where we made use of the fact that $i_{\widetilde{u}}\mathrm{v} = -1$. But from (42) we have $i_\sigma \mathrm{v} = i_\sigma i_\mathrm{v} \widehat{d}$ and from (43) $\beta = \Phi - \Psi = -i_{\widetilde{\sigma}} i_{\widetilde{u}} \star \hat{\mathrm{k}} - i_{\widetilde{u}} \widehat{d}$. By its definition Λ can be expressed as

$$\Lambda \equiv \star(R_{ab} \wedge \sigma \wedge \Psi) \star (e^a \wedge e^b). \tag{50}$$

Since $\widehat{f} = \frac{m}{2} \star (R_{ab} \wedge \sigma \wedge u)(e^a \wedge e^b)$ the last equation simply becomes $\Lambda = \frac{2}{m} \star \widehat{f}$. Back in (48) we can write

$$\mathrm{v}^c - u^c - \beta^c = \mathrm{v}^c(i_{\widetilde{\sigma}} i_{\widetilde{u}} \Lambda) + \Lambda^{cs} u_s(-i_{\widetilde{\sigma}} i_{\widetilde{u}} \widehat{d}) + \Lambda^{cs}\sigma_s. \tag{51}$$

So,

$$(1 - i_{\widetilde{\sigma}} i_{\widetilde{u}} \Lambda)\mathrm{v}^c = u^c + \beta^c + \Lambda^{cs}\sigma_s - \Lambda^{cs} u_s i_{\widetilde{\sigma}} i_{\widetilde{u}} \widehat{d} \tag{52}$$

or

$$(1 - i_{\widetilde{\sigma}} i_{\widetilde{u}} \Lambda)\mathrm{v}^c = u^c + i^c i_{\widetilde{\sigma}} i_{\widetilde{u}} \star \hat{\mathrm{k}} - i^c i_{\widetilde{u}} \widehat{d} + \Lambda^{cs} i_\sigma \sigma + \Lambda^{cs} i_s u(i_{\widetilde{\sigma}} i_{\widetilde{u}} \widehat{d}). \tag{53}$$

Now remember that $i_{\widetilde{\sigma}} i_{\widetilde{u}} \Lambda = i_{\widetilde{u}} i_{\widetilde{\sigma}} \widehat{\widehat{R}}_{rs} u^r \sigma^s$ where $\Lambda \equiv \frac{1}{2}\Lambda^{rs} e_r \wedge e_s$. Therefore,

$$\Lambda^{cs}\sigma_s e_s = (i_{\widetilde{u}} i_{\widetilde{\sigma}} \widehat{\widehat{R}}_{rs})\sigma_s e_c \tag{54}$$

and

$$i_{\widetilde{\sigma}}\Lambda = \Lambda^{rs}\sigma_r e_r = -\Lambda^{sr}\sigma_r e_s \tag{55}$$

$$\Rightarrow \Lambda^{cs}\sigma_s e_c \equiv -i_{\widetilde{\sigma}}\Lambda. \tag{56}$$

Similarly, we can show that

$$\Lambda^{cs}u_s e_c \equiv -i_{\widetilde{u}}\Lambda. \tag{57}$$

Now let $\beta = \beta^c e_c = i_{\widetilde{u}}\widehat{d} - i_{\widetilde{\sigma}}i_{\widetilde{u}}\hat{\mathrm{k}}$. So back in (53) we can write,

$$\mathrm{v} = \frac{u - i_\sigma\Lambda}{(1 + i_{\widetilde{u}}i_{\widetilde{\sigma}}\Lambda)} + \frac{(i_{\widetilde{\sigma}}i_{\widetilde{u}}\widehat{d})i_{\widetilde{u}}\Lambda - i_{\widetilde{u}}\widehat{d} - i_{\widetilde{\sigma}}i_{\widetilde{u}} \star \hat{k}}{(1 + i_{\widetilde{u}}i_{\widetilde{\sigma}}\Lambda)}, \tag{58}$$

where $\Lambda = \frac{1}{2}(i_{\widetilde{u}}i_{\widetilde{\sigma}}\widehat{\widehat{R}}_{rs})e^r \wedge e^s$, $\widehat{\widehat{R}}_{pq} = \star\widehat{R}_{pq}$, $\widehat{R}_{pq} = \frac{1}{2}\epsilon^{ab}_{pq}R_{ab}$, $\widehat{\mathrm{d}} = \frac{1}{2}\star(1 + u \wedge i_{\widetilde{u}}) \star \mathrm{d}$, $\widehat{\mathrm{k}} = (1 + ui_{\widetilde{u}})\mathrm{k}$, $\mathrm{k} = \mathcal{K}/m'$ and $\mathrm{d} = \mathcal{D}/m'$. Note that Eq. (58) gives explicitly the velocity of a test spinning particle, v in terms of u, σ, d and k, provided a spacetime R_{ab}. We argue that the differential algebraic system of Eqs. (12), (14), (29) and (58) is adequate to define the motion of a spinning particle in a given arbitrary background metric.

3. CONSTANTS OF MOTION

As we have already seen in the previous section

$$\dot{s} = 2p \wedge \mathrm{v} + m'd = 2m'u \wedge \mathrm{v} + m'd. \tag{59}$$

Since $p \wedge \star s = 0$, we can write

$$s \wedge \star\dot{s} = 2s \wedge \star(p \wedge \mathrm{v}) + m's \wedge \star\mathrm{d} = 2(p \wedge \mathrm{v}) \wedge \star s + m's \wedge \star d = m's \wedge \star d, \tag{60}$$

and

$$(s \wedge \star s)^{\cdot} = 2s \wedge \star\dot{s} = 2m's \wedge \star d. \tag{61}$$

Now define the 0-form λ as

$$\lambda = \frac{1}{2}\mathbf{G}(l, l), \tag{62}$$

then

$$\dot{\lambda} = \mathbf{G}(\dot{l}, l) = i_{\widetilde{l}}\dot{l} \equiv m'^2 i_{\widetilde{\sigma}}\dot{\sigma}, \tag{63}$$

where $l = m'\sigma$. Also recall that $s = 2\star(l \wedge u)$ and the transformation law for σ is given by (29). We next apply $i_{\widetilde{\sigma}}$ on both sides of Eq. (29) and write,

$$i_{\widetilde{\sigma}}(\dot{m'}\sigma) + i_{\widetilde{\sigma}}[m'(i_{\widetilde{u}}\dot{\sigma} \wedge u + \dot{\sigma})] = -\frac{1}{2}i_{\widetilde{\sigma}}i_{\widetilde{u}}(m' \star d). \tag{64}$$

Since $i_{\widetilde{\sigma}}\sigma = 0$ and $i_{\widetilde{\sigma}}\dot{m}' = 0$ the first term vanishes and we simply get

$$i_{\widetilde{\sigma}}(i_{\widetilde{u}}\dot{\sigma}) \wedge u - i_{\widetilde{u}}\dot{\sigma} \wedge (i_{\widetilde{\sigma}}u) + i_{\widetilde{\sigma}}\dot{\sigma} = -\frac{1}{2}i_{\widetilde{\sigma}}i_{\widetilde{u}}(\star d). \tag{65}$$

Remember that $i_{\widetilde{u}}\sigma = i_{\widetilde{\sigma}}u = 0$, so the second term becomes zero. Equation (65) can be subsequently written as

$$i_{\widetilde{\sigma}}(i_{\widetilde{u}}\dot{\sigma}) \wedge u + i_{\widetilde{\sigma}}\dot{\sigma} = -\frac{1}{2}i_{\widetilde{\sigma}}i_{\widetilde{u}}(\star d). \tag{66}$$

If we now use the fact that $i_{\widetilde{\sigma}}i_{\widetilde{u}} = -i_{\widetilde{u}}i_{\widetilde{\sigma}}$ we can write

$$i_{\widetilde{\sigma}}\dot{\sigma} = -\frac{1}{2}\frac{i_{\widetilde{\sigma}}i_{\widetilde{u}}(\star d)}{1 - ui_{\widetilde{u}}}. \tag{67}$$

If we substitute (67) in (63) we get

$$\dot{\lambda} = m'^2 i_{\widetilde{\sigma}}\dot{\sigma} = -\frac{m'^2}{2}\frac{i_{\widetilde{\sigma}}i_{\widetilde{u}}(\star d)}{1 - ui_{\widetilde{u}}}. \tag{68}$$

Therefore, $\|l\| = 2\lambda$ is a constant of motion iff $i_{\widetilde{\sigma}}i_{\widetilde{u}}(\star d) = 0$.

Next, we compute the scalar m' defined by $p = m'u$ and $\mathbf{g}(u, u) = -1$. As before, we can write

$$-m'^2 = \mathbf{G}(p, p) \equiv i_{\widetilde{p}}p. \tag{69}$$

We know that

$$\dot{s} = 2p \wedge \mathrm{v} + \mathcal{D}. \tag{70}$$

If we apply $i_{\widetilde{p}}$ we then have

$$\begin{aligned} i_{\widetilde{p}}\dot{s} &= 2i_{\widetilde{p}}(p \wedge \mathrm{v}) + i_{\widetilde{p}}\mathcal{D} \\ &= 2i_{\widetilde{p}}p \wedge \mathrm{v} - 2p \wedge i_{\widetilde{p}}\mathrm{v} + i_{\widetilde{p}}\mathcal{D} \\ &= -2m'^2\mathrm{v} - 2p \wedge i_{\widetilde{p}}\mathrm{v} + i_{\widetilde{p}}\mathcal{D}, \end{aligned} \tag{71}$$

where we have made use of (69). If $i_{\widetilde{p}}\mathrm{v} \neq 0$, then we can write

$$p = -\frac{1}{2i_{\widetilde{p}}\mathrm{v}}(i_{\widetilde{p}}\dot{s} + m'^2\mathrm{v} - i_{\widetilde{p}}\mathcal{D}). \tag{72}$$

Also, from (69) we have

$$-(m'^2)^{\cdot} = 2\mathbf{G}(\dot{p}, p) = 2i_{\widetilde{p}}\dot{p} = 2i_{\widetilde{\dot{p}}}p. \tag{73}$$

If we now substitute (72) in (73) we get

$$-(m'^2)^{\cdot} = \frac{1}{i_{\widetilde{p}}\mathrm{v}}(i_{\widetilde{\dot{p}}}i_{\widetilde{p}}\mathcal{D} - m'^2 i_{\mathrm{v}}\mathcal{K} - i_{\widetilde{\dot{p}}}i_{\widetilde{p}}\dot{s}), \tag{74}$$

where we have made use of $\dot{p} = i_{\tilde{v}} f + \mathcal{K}$. Now since $i_{\tilde{P}} s = 0$, i.e $s(P, -) = 0$, by differentiating we get

$$\dot{s}(P, -) + s(\dot{P}, -) = 0, \tag{75}$$

or

$$i_{\tilde{P}} \dot{s} = -i_{\tilde{\dot{P}}} s. \tag{76}$$

Therefore, $i_{\tilde{P}} i_{\tilde{P}} \dot{s} = 0$ and (74) becomes,

$$(m'^2)^{\cdot} = \frac{1}{i_{\tilde{P}} \mathrm{v}} (m'^2 i_{\mathrm{v}} \mathcal{K} - i_{\tilde{P}} i_{\tilde{P}} \mathcal{D}). \tag{77}$$

Then, if $i_{\tilde{P}} \mathrm{v} \neq 0$, m'^2 is a constant of motion if the numerator of (77) is zero.

4. BRANS-DICKE THEORY

As an application we consider the simplest modification of General Relativity. In 1961 Brans and Dicke suggested a change in Einstein's theory by introducing an additional scalar field with a specific gravitational coupling to matter via the space-time metric (Brans and Dicke, 1961; Dicke, 1962). They postulated that this scalar field determines the value of the locally varying gravitational coupling constant and that $\phi \sim G^{-1}$. They also argued that classical tests of gravitational theories might have failed to detect experimentally this new scalar component. The dispute is still open and there are recent proposals about new detectable implications (Dereli and Tucker, 2001).

In their original paper (Brans and Dicke, 1961) they assumed that the motion of a test particle follows a Levi-Civita auto-parallel associated with the metric derived from the Brans-Dicke field equations. This assumption could also stand when the scalar field varies in spacetime. In a later work Dirac demonstrated that, in a Weyl invariant generalisation, it is more natural to generate the motion of a test particle from a Weyl invariant action principle (Dirac, 1973). In general, such a motion is different from the Brans-Dicke, Levi-Civita auto-parallel and it turns out that even neutral test particles would follow auto-parallels of a connection with torsion (Dereli and Tucker, 2001). It has been shown that the Brans-Dicke theory can be reformulated as a field theory on a spacetime with dynamic torsion T determined by the gradient of the Brans-Dicke scalar field:

$$T = e^a \otimes \frac{d\phi}{2\phi} \otimes X_a - \frac{d\phi}{2\phi} \otimes e^a \otimes X_a, \tag{78}$$

where e^a is any coframe and X_a its dual (Dereli and Tucker, 1982). In differential forms notation the above equation can be expressed as

$$T^a = e^a \wedge \frac{d\phi}{2\phi} \tag{79}$$

Now recall that the general connection is defined as (Benn and Tucker, 1987)

$$\nabla_{X_a} X_b \equiv \omega^c{}_b(X_a)X_c, \tag{80}$$

where $\omega^c{}_b$ are the connection 1-form and $\{e^a\}$ the co-frame dual to $\{X_a\}$. The connection 1-forms can be decomposed into its Riemannian and non-Riemannian part ${}^0\omega^a{}_b$ and $K^a{}_b$ respectively,

$$\omega^a{}_b = {}^0\omega^a{}_b + K^a{}_b, \tag{81}$$

where ${}^0\omega^a{}_b$ are the Levi-Civita connection 1-forms. It can be shown that (Tucker and Wang, 1995)

$$2{}^0\omega^a{}_b = (g_{ac}i_b - g_{bc}i_a + e_c i_a i_b)de^c + (i_b dg_{ac} - i_a dg_{bc})e^c + dg_{ab} \tag{82}$$

and

$$2K^a{}_b = i_a T_b - i_b T_a - i_a i_b T_c e^c. \tag{83}$$

In turn, the curvature 2-forms are given by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{84}$$

and can be decomposed as

$$R^a{}_b = {}^0R^a{}_b + {}^0DK^a{}_b + K^a{}_c \wedge K^c{}_b, \tag{85}$$

where

$${}^0DK^a{}_b = dK^a{}_b +^0 \omega^a{}_c \wedge K^c{}_b + K^a{}_c \wedge^0 \omega^c{}_b, \tag{86}$$

and ${}^0R^a{}_b$ is the Riemannian part of the curvature 2-forms (Dereli and Tucker, 1982).

Let us consider now the differential algebraic system of Eqs. (12), (14), (29) and (58) in a background metric $\mathbf{g}$ and a scalar field ϕ. The problem is further simplified when the drive 1-form $\mathcal{K}$ and 2-form $\mathcal{D}$ are set equal to zero (dipole approximation). In this case the equations under consideration become,

$$\nabla_V^{(\mathbf{g},T)} P = \widetilde{i_V f}, \tag{87}$$

$$i_P s = 0, \tag{88}$$

$$(1 + u \wedge i_U)\nabla_V^{(\mathbf{g},T)}\sigma = 0 \tag{89}$$

and

$$\mathrm{v} = \frac{u - i_\Sigma \Lambda}{1 + i_U i_\Sigma \Lambda}, \tag{90}$$

where $f = -\frac{1}{4} \star (R_{ab} \wedge s)e^a \wedge e^b$ and $\Lambda = \frac{1}{2}[i_{\tilde{u}} i_{\tilde{\sigma}} \star (\frac{1}{2}\epsilon^{ab}_{pq} R_{ab})]e^r \wedge e^s$. Also, the action of the connection $\nabla^{(\mathbf{g},T)}$ is defined by (80). We must stress that the curvature in the above equations can involve non-Riemannian contributions. In this way modifications to geodesics due to both scalar fields and particle's spin can be included.

For the Brans-Dicke theory the 1-forms $K_{ab} = -K_{ba}$ can be expressed in terms of the scalar field ϕ and it has been shown that (Dereli and Tucker, 1982)

$$K^a{}_b = \frac{1}{2\phi}(e^a i_b d\phi - e_b i^a d\phi). \tag{91}$$

If we use Eqs. (86) and (91) in (85), after some algebra one can show that

$$\begin{aligned} R_{ab} = {}^0R_{ab} - \frac{d\phi}{2\phi^2} \wedge [i_b(d\phi)e_a - i_a(d\phi)e_b] + \frac{1}{2\phi} d[i_b(d\phi)e_a - i_a(d\phi)e_b] \\ + \frac{1}{2\phi}[i_b(d\phi)^0\omega_{ac} \wedge e^c - i_c(d\phi)^0\omega_{ac} \wedge e_b] + \frac{1}{2\phi}[i_c(d\phi)e_a \wedge^0 \omega^c{}_b \\ - i_a(d\phi)e_c \wedge^0 \omega^c{}_b] + \frac{1}{4\phi^4}[i_c(d\phi)i_b(d\phi)e_a \wedge e^c - i_c(d\phi)i^c(d\phi)e_a \wedge e_b \\ + i_a(d\phi)i^c(d\phi)e_c \wedge e_b]. \end{aligned} \tag{92}$$

It is well known that any theory written in terms of a geometry with non-trivial ($\mathbf{g}$, T) and ϕ can be reformulated in terms of a geometry with either ($\mathbf{g}$, 0) or ($\phi\mathbf{g}$, 0) (Dereli and Tucker, 2001). Moreover it is also true that

$$\nabla^{(\mathbf{g},T)} = \nabla^{(\tilde{\mathbf{g}},0)} - \frac{d\phi}{2\phi}, \tag{93}$$

where $\tilde{\mathbf{g}} = \phi\mathbf{g}$ with the above choice of T and both connections metric compatible. An interesting reformulation of the Brans-Dicke theory occurs by applying a transformation

$$\tilde{e}^a = \left(\frac{\phi}{\phi_0}\right)^{\frac{1}{2}} e^a \tag{94}$$

for the orthonormal frames, where ϕ_0 is a constant. The new coframe fields $\tilde{e}^a$ become orthonormal with respect to the spacetime metric $\tilde{\mathbf{g}}$ such that

$$\tilde{\mathbf{g}} = \left(\frac{\phi}{\phi_0}\right)\mathbf{g}. \tag{95}$$

In this case the MPD equations become

$$\nabla_V^{(\tilde{\mathbf{g}},0)} P = \widetilde{i_V f}, \tag{96}$$

$$i_P s = 0, \tag{97}$$

$$(1 + u \wedge i_U)\nabla_V^{(\tilde{\mathbf{g}},0)}\sigma = 0 \tag{98}$$

and

$$\mathrm{v} = \frac{u - i_{\Sigma}\Lambda}{1 + i_U i_{\Sigma}\Lambda}, \tag{99}$$

where now $f = -\frac{1}{4} \star ({}^0R'_{ab} \wedge s)e^a \wedge e^b$ and $\Lambda = \frac{1}{2}[i_{\tilde{u}} i_{\tilde{\sigma}} \star (\frac{1}{2}\epsilon^{ab}_{pq}({}^0R'_{ab}))]e^r \wedge e^s$. Note that ${}^0R'_{ab}$ are curvature 2-forms associated with the new metric $\tilde{\mathbf{g}}$.

5. CONCLUSIONS

To sum up, we argue that the differential algebraic system of Eqs. (12), (14), (29) and (58) is adequate to define the motion of a spinning particle in a given arbitrary background metric. The problem is further simplified in the case of dipole approximation, when the drive 1-form $\mathcal{K}$ and 2-form $\mathcal{D}$ are set equal to zero . Then, the equations under consideration become,

$$\dot{P} = \widetilde{i_V f}, \tag{100}$$

$$i_P s = 0, \tag{101}$$

$$(1 + u \wedge i_U)\dot{\sigma} = 0 \tag{102}$$

and

$$\mathrm{v} = \frac{u - i_{\Sigma}\Lambda}{1 + i_U i_{\Sigma}\Lambda}, \tag{103}$$

with $f = -\frac{1}{4} \star (R_{ab} \wedge s)e^a \wedge e^b$ and $\Lambda = \frac{1}{2}[i_{\tilde{u}} i_{\tilde{\sigma}} \star (\frac{1}{2}\epsilon^{ab}_{pq} R_{ab})]e^r \wedge e^s$. We must stress that the curvature in the above equations can involve non-Riemannian contributions. Hence, the MPD equations can be analysed in background metrics more general than the ones considered by the theory of General Relativity. In this way modifications to geodesics due to both scalar fields and particle's spin can be included.

Finally, we should point out that Eq. (77) in the dipole approximation gives $(m'^2)^{\cdot} = 0$. Therefore, we recover the familiar result that the mass of the spinning particle is preserved along the particle's worldline. However, the notion of particle's mass is not trivial in these equations and it is only a constant of the motion if certain properties of the multiples are zero. This outcome questions the equivalence principle on which Einstein's theory is founded.

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